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Resonant sloshing near a critical depth

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Oscillations of a tank at a near-resonant frequency have been shown to produce a response which changes from a 'hard-spring' to a 'soft-spring' response as the depth passes through a critical value. This paper investigates the transition region and it is shown, using a symbolic manipulator, that in fact the large-amplitude response is that of a soft spring on either side of this critical depth.

1. Introduction

The problem of small horizontal oscillations of a rectangular tank of water (with depth comparable to width) has been considered in some detail. Moiseyev (1958) considered how the interaction of the nonlinearity affected the response for a tank oscillated near its fundamental frequency. Ockendon & Ockendon (1973) investigated the problem using asymptotic expansions for the velocity potential and free surface profile. They showed that the response is exactly the same as that of the undamped Duffing equation and changes from 'hard-spring' (increasing amplitude with increasing frequency) to 'soft-spring' (decreasing amplitude with increasing frequency) behaviour (see, for instance, Jordan & Smith 1977, § 5.6) as the depth passes through a certain value. This result was also noticed by Tadjbakhsh & Keller (1960) who considered the possible frequencies of standing waves in a tank, and Fultz (1962) provided experimental results to confirm this change in behaviour. What happens at the critical depth where the response changes from being a 'hard-spring' to 'soft-spring' is the problem that will be considered here. This special case was hypothesized by Ockendon & Ockendon (1973) and it is the aim here, through the use of a symbolic manipulator, to elucidate the response diagram for depths in the vicinity of this critical value.

2. Problem formulation

Following Ockendon & Ockendon (1973) inviscid, irrotational two-dimensional fluid motion of mean depth hL in a tank of length πL is considered. The tank is given a horizontal oscillation of amplitude εL ($\varepsilon \ll 1$) and frequency ω. The equations of motion are

$$\nabla^2\phi = 0, \tag{2.1}$$

$$\phi_x = \sin t \quad \text{on} \quad x = -\varepsilon\cos t, \pi - \varepsilon\cos t, \tag{2.2}$$

$$\phi_z = 0 \quad \text{on} \quad z = -h, \tag{2.3}$$

$$\nabla\phi(x,z,t) = \nabla\phi(x,z,t+2\pi), \tag{2.4}$$

$$\int_0^\pi \eta \, \mathrm{d}x = 0, \tag{2.5}$$

and, on $z = \varepsilon\eta$,

$$\phi_z = \eta_t + \varepsilon\phi_x\eta_x, \tag{2.6}$$

$$\eta + \frac{L\omega^2}{g}\left[\phi_t + \tfrac{1}{2}\varepsilon(\phi_x^2 + \phi_z^2)\right] = 0 \tag{2.7}$$

where the dimensional velocity potential is $\varepsilon L^2\omega\phi$ and the dimensional free surface height is $\varepsilon L\eta$. To consider oscillations near the fundamental frequency write $L\omega^2 = (1+\delta)g\tanh h$ so that (2.7) becomes

$$\eta + (1+\delta)\tanh h\left[\phi_t + \tfrac{1}{2}\varepsilon(\phi_x^2 + \phi_z^2)\right] = 0, \tag{2.8}$$

and δ (the detuning) measures how close the frequency of the oscillations of the tank are to the fundamental frequency. Away from resonance, when δ is large, the solution can be found by considering the linearized form of (2.1)–(2.6) and (2.8) which leads to the solution

$$\phi \sim \frac{4}{\delta\pi\cosh h}\cos x\cosh(z+h)\sin t. \tag{2.9}$$

Hence as $\delta \to 0$ the response is $O(\varepsilon/\delta)$ which blows up at exact resonance. In this case the nonlinear terms are no longer negligible and in the regime $\delta\varepsilon^{-2/3} \sim O(1)$ we pose the expansion $\phi \sim \varepsilon^{-2/3}\phi_0 + \varepsilon^{-1/3}\phi_1 + \phi_2 + \cdots$ with a similar expansion for η. Solving the resulting problems for ϕ_0 and ϕ_1 and using a solvability condition on the problem for ϕ_2 leads to

$$\phi \sim \varepsilon^{-2/3}\bar{A}\cos x\cosh(z+h)\sin t,$$

where $\bar{A}$ satisfies

$$\frac{4}{\pi}\tanh h = \delta\varepsilon^{-2/3}\bar{A}\sinh h + H(h)\bar{A}^3 \tag{2.10}$$

and

$$H(h) = -\tfrac{1}{32}\operatorname{sech}^2 h\,\operatorname{cosech} h\,(9 + 15\sinh^2 h - 8\sinh^6 h). \tag{2.11}$$

So the response, as shown in figure 1, is the same as the undamped Duffing equation and looks like a 'soft-spring' for $H > 0$ and a 'hard-spring' for $H < 0$. When $H(h) \equiv 0$ (at $h = h_0 = 1.06$) the expansion breaks down as $\bar{A}$ grows large in a narrow detuning band near the origin.

3. Problem near the critical depth

As noticed by Ockendon & Ockendon (1973) as h gets closer to h_0 it is necessary to rescale the variables and pose new expansions for ϕ and η. To deduce the correct scalings in this case put

$$h = h_0 + \varepsilon^\alpha h_1, \tag{3.1}$$

where α is a positive constant to be found. Substituting this into (2.10) means that $\bar{A}^3\varepsilon^\alpha \sim O(1)$ together with $\delta\varepsilon^{-2/3}\bar{A} \sim O(1)$ for all the terms to balance. Hence $\bar{A} = \varepsilon^{-\alpha/3}A$ and $\delta \sim O(\varepsilon^{(\alpha+2)/3})$ so that

$$\phi \sim \varepsilon^{-(\alpha+2)/3}A\cos x\cosh(z+h)\sin t. \tag{3.2}$$

For any expansion of ϕ and η it is necessary that the nonlinear terms generate $\cos x\sin t$ terms at $O(1)$ so that the boundary conditions (2.2) can be fitted. This

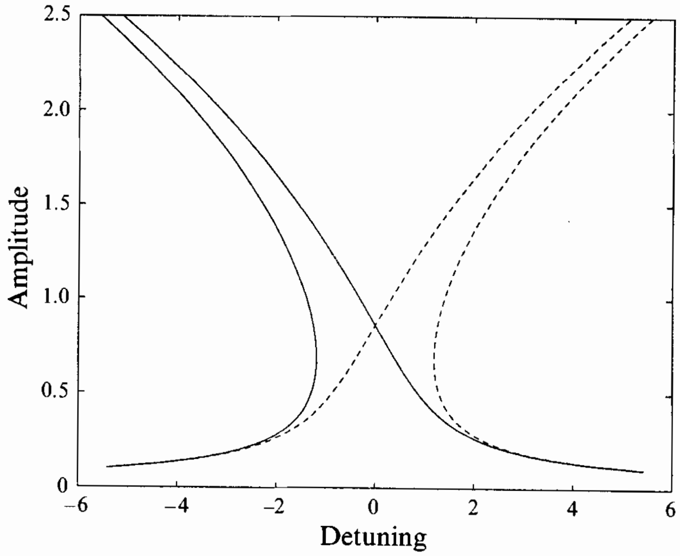

FIGURE 1. Classic Duffing-like response (2.10) showing how amplitude $|\bar{A}|$ varies with detuning $\delta\varepsilon^{-2/3}$: ——, $H > 0$; - - - -, $H < 0$.

means that the $O(1)$ problem must involve *odd* powers of ϕ_0, in other words the first term in an expansion for ϕ must be of the form $\varepsilon^{-2n/(2n+1)}$ $(n = 0, 1, 2, \ldots)$. Coupling this with (3.2) shows that the simplest system will be when $(\alpha + 2)/3 = 4/5$ or when $\alpha = 2/5$. Hence the scalings are

$$h = h_0 + \varepsilon^{2/5} h_1 \quad \text{and} \quad \delta\varepsilon^{-4/5} \sim O(1), \tag{3.3}$$

where h_1 is $O(1)$, together with the expansions

$$\phi \sim \varepsilon^{-4/5}\phi_0 + \varepsilon^{-3/5}\phi_1 + \varepsilon^{-2/5}\phi_2 + \varepsilon^{-1/5}\phi_3 + \phi_4 + \cdots \tag{3.4}$$

and

$$\eta \sim \varepsilon^{-4/5}\eta_0 + \varepsilon^{-3/5}\eta_1 + \varepsilon^{-2/5}\eta_2 + \varepsilon^{-1/5}\eta_3 + \eta_4 + \cdots, \tag{3.5}$$

so that near h_0 the amplitude of the response grows from $O(\varepsilon^{1/3})$ to $O(\varepsilon^{1/5})$. Substituting (3.3), (3.4) and (3.5) into equations (2.1)–(2.6) and (2.8) and comparing $O(\varepsilon^{-4/5})$ terms leads to the same first-order problem as before, and so the leading-order solutions are

$$\phi_0 = A \cos x \cosh (z + h) \sin t, \tag{3.6}$$

$$\eta_0 = -A \sinh h \cos x \cos t \tag{3.7}$$

but now to determine A it is necessary to find a solvability condition on the problem for ϕ_4 and η_4. To do this the $O(\varepsilon^{-3/5})$, $O(\varepsilon^{-2/5})$ and $O(\varepsilon^{-1/5})$ terms from the equations must be found and solved to determine ϕ_i and η_i, $i = 1, 2, 3$. (The results for ϕ_i and η_i are included in Appendix A.) Next considering the $O(1)$ terms from (2.1)–(2.4), (2.6) and (2.8), it is found that

$$\nabla^2 \phi_4 = 0, \tag{3.8}$$

$$\phi_{4x} = \sin t \quad \text{on} \quad x = 0, \pi, \tag{3.9}$$

$$\phi_{4z} = h_1 \phi_{2zz} \quad \text{on} \quad z = -h_0, \tag{3.10}$$

$$\phi_{4z} + \tanh h_0 \phi_{4tt} = F(x, t) \quad \text{on} \quad z = 0, \tag{3.11}$$

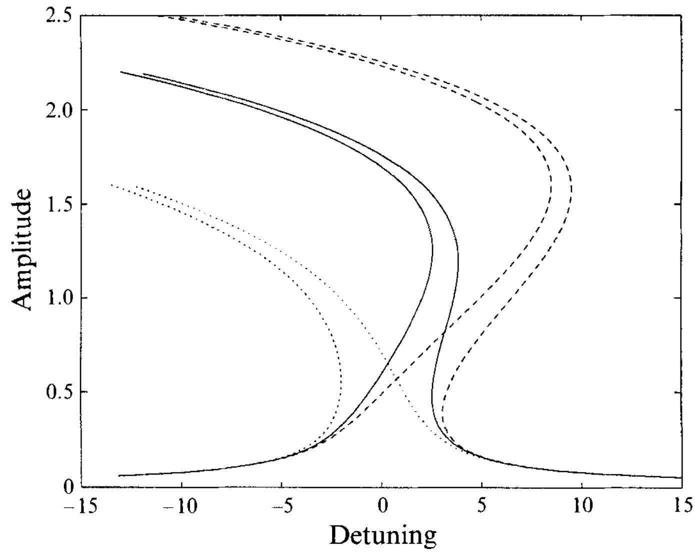

FIGURE 2. Response based on equation (3.14) – a plot of amplitude $|A|$ against detuning $\delta\varepsilon^{-4/5}$ in three cases: $\cdots\cdots$, $h_1 = 1$; ——, $h_1 = -2.98$; - - - -, $h_1 = -5$.

$$\nabla\phi_4(x,z,t) = \nabla\phi_4(x,z,t+2\pi), \tag{3.12}$$

where F is a combination of ϕ_i and η_i $(i = 0,1,2,3)$ given explicitly in Appendix B. Using the Fredholm Alternative on (3.8)–(3.12) gives the solvability condition

$$\frac{2}{\pi^2}\int_0^{2\pi}\int_0^{\pi} F(x,t)\cos x \sin t \mathrm{d}x\mathrm{d}t - 0.62h_1C = \frac{4}{\pi}\tanh h_0, \tag{3.13}$$

where C comes from the expression for ϕ_2 (see Appendix A). Substituting the expressions for ϕ_i and η_i, $i = 0,1,2,3$, into F and using (3.13) finally yields

$$\delta\varepsilon^{-4/5}A\sinh h_0 + A^3h_1H'(h_0) + 1.80A^5 = \frac{4}{\pi}\tanh h_0. \tag{3.14}$$

Figure 2 shows plots of the response $|A|$ against $\delta\varepsilon^{-4/5}$ for h_1 ranging from 1 to -5. It is clear that close to either side of this critical depth the large-amplitude response is like that of a 'soft-spring'. As h_1 decreases through -2.98 the number of solutions changes from 3 to 5 and the 'hard-spring' response begins to emerge. When h_1 is very large and negative the large-amplitude branches move off to infinity and the response will look like figure 1 for $H < 0$. This can also be seen by taking the limit of (2.10) as $h \to h_0$. To do this write $h = h_0 + \varepsilon^{2/5}h_1$ in (2.10) to get

$$\delta\varepsilon^{-2/3}\bar{A}\sinh h_0 + \varepsilon^{2/5}\bar{A}^3h_1H'(h_0) + o(\varepsilon^{2/5}) = \frac{4}{\pi}\tanh h_0$$

which matches (3.14) as h_1 grows to $O(\varepsilon^{2/5})$ with $A = \varepsilon^{2/15}\bar{A}$. We also note that the coefficient of the quintic term in (3.14) never vanishes so that this response is valid as $h_1 \to 0$.

Finally we note that we can systematically write down a uniformly valid response for all depths and detunings if we consider

$$\Phi = \omega^2L[A_\varepsilon\cos x\cosh(z+h)\sin t + O(\varepsilon^{2/n})],$$
$$\xi = L[-A_\varepsilon\sinh h\cos x\cos t + O(\varepsilon^{2/n})]$$

where Φ and ξ are the dimensional velocity potential and free surface respectively, $A_\varepsilon = \varepsilon^{1/n}A$ and n takes the values 1, 2 or 3 depending on the detuning range. It is clear that the final response can be written independently of n in the form

$$\frac{4\varepsilon}{\pi}\tanh h = \delta A_\varepsilon \sinh h + A_\varepsilon^3 H(h) + A_\varepsilon^5 H_2(h) + O(\delta^2 A_\varepsilon) \tag{3.15}$$

where $H_2(h_0) = 1.80$. Clearly (3.15) encompasses (2.9), (2.10) and (3.14).

4. Conclusions

The response near the critical depth h_0 has been found and is valid for all h_1 showing that if $h_1 < 0$ there is a large-amplitude response which was not apparent from the earlier analysis of Ockendon & Ockendon (1973). The response for all non-shallow depths can now be given and a unifying statement (3.15) has been found. The stability of the different branches has not been determined, but it is likely that, as in Duffing's equation, the stability will change at each vertical tangent of the response diagram.

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Appendix A. Expressions for ϕ_i and η_i $(i = 1, 2, 3)$

The results for ϕ_i are included for completeness. To derive these results is extremely long-winded and it is noted that the use of the symbolic manipulator MAPLE was used to simplify matters.

$$\begin{aligned}
\phi_1 &= B\mathscr{C}_1 \sin t + A^2(0.46 - 0.12\mathscr{C}_2)\sin 2t - (A^2/8)t,\\
\phi_2 &= C\mathscr{C}_1 \sin t + AB(0.93 - 0.23\mathscr{C}_2)\sin 2t + A^3(-0.59\mathscr{C}_1 + 0.004\mathscr{C}_3)\sin 3t\\
&\quad +0.01A^3\mathscr{C}_3 \sin t - (AB/4)t + Ah_1 \cos x \sinh(z + h_0)\sin t,\\
\phi_3 &= D\mathscr{C}_1 \sin t + ([0.26A^4 - 0.23AC + 0.30h_1A^2 - 0.12B^2]\mathscr{C}_2\\
&\quad +0.004A^4\mathscr{C}_4 + [1.02h_1A^2 + 0.46B^2 + 0.60A^4 + 0.93AC])\sin 2t\\
&\quad +A^2B(-1.76\mathscr{C}_1 + 0.01\mathscr{C}_3)\sin 3t + A^4(0.001\mathscr{C}_4 + 0.15\mathscr{C}_2 + 0.13)\sin 4t\\
&\quad +h_1[B\cos x \sinh(z + h_0)\sin t - 0.23\cos 2x \sinh 2(z + h_0)\sin 2t]\\
&\quad +0.03A^2B\mathscr{C}_3 \sin t
\end{aligned}$$

where B, C and D are arbitrary constants and

$$\mathscr{C}_n = \cos nx \cosh n(z + h_0), \quad n = 1, 2, 3, 4.$$

The η_i $(i = 1, 2, 3)$ can then be found from equation (2.8).

Appendix B. Expression for $F(x, t)$

The function $F(x, t)$ is given by

$$\begin{aligned}
F(x,t) = F_1(x,0,t) - \tanh h_0 \frac{\partial}{\partial t}F_2(x,0,t) + h_1(\tanh^2 h_0 - 1)\frac{\partial}{\partial t}F_3(x,0,t)\\
- \tanh h_0(\delta\varepsilon^{-4/5} + h_1^2(\tanh^2 h_0 - 1))\frac{\partial^2 \phi_0}{\partial t^2}(x,0,t)
\end{aligned}$$

where

$$
\begin{aligned}
F_1(x,z,t) = {} & \eta_2\phi_{0xz}\eta_{0x} + \eta_0\eta_1\phi_{0x}\eta_{0x} + \tfrac{1}{6}\eta_0^3\phi_{0xz}\eta_{0x} + \eta_1\phi_{0xz}\eta_{1x} + \eta_1\phi_{1xz}\eta_{0x} \\
& + \tfrac{1}{2}\eta_0^2\phi_{0x}\eta_{1x} + \tfrac{1}{2}\eta_0^2\phi_{1xzz}\eta_{0x} + \eta_0\phi_{0xz}\eta_{2x} + \eta_0\phi_{2xz}\eta_{0x} + \eta_0\phi_{1xz}\eta_{1x} \\
& + \phi_{3x}\eta_{0x} + \phi_{0x}\eta_{3x} + \phi_{1x}\eta_{2x} + \phi_{2x}\eta_{1x} - \eta_3\phi_0 - \eta_0\eta_2\phi_{0z} - \tfrac{1}{2}\eta_1^2\phi_{0z} \\
& - \tfrac{1}{2}\eta_0^2\eta_1\phi_0 - \tfrac{1}{24}\eta_0^4\phi_{0z} - \eta_2\phi_{1zz} - \eta_0\eta_1\phi_{1zzz} - \tfrac{1}{6}\eta_0^3\phi_{1zzzz} \\
& - \eta_1\phi_{2zz} - \tfrac{1}{2}\eta_0^2\phi_{2zzz} - \eta_0\phi_{3zz}, \\
F_2(x,z,t) = {} & \eta_3\phi_{0tz} + \eta_0\eta_2\phi_{0t} + \tfrac{1}{2}\eta_1^2\phi_{0t} + \tfrac{1}{2}\eta_0^2\eta_1\phi_{0tz} + \tfrac{1}{24}\eta_0^4\phi_{0t} + \eta_2\phi_{1tz} \\
& \eta_0\eta_1\phi_{1tzz} + \tfrac{1}{6}\eta_0^3\phi_{1tzzz} + \eta_1\phi_{2tz} + \tfrac{1}{2}\eta_0^2\phi_{2tzz} + \eta_0\phi_{3tz} \\
& + \eta_2\phi_{0x}\phi_{0xz} + \eta_0\eta_1\phi_{0xz}^2 + \eta_0\eta_1\phi_{0x}^2 + \tfrac{2}{3}\eta_0^3\phi_{0xz}\phi_{0x} \\
& + \eta_1\phi_{0xz}\phi_{1x} + \eta_1\phi_{0x}\phi_{1xz} + \tfrac{1}{2}\eta_0^2\phi_{0x}\phi_{1x} + \eta_0^2\phi_{0xz}\phi_{1xz} \\
& + \tfrac{1}{2}\eta_0^2\phi_{0x}\phi_{1xzz} + \eta_0\phi_{1x}\phi_{1xz} + \eta_0\phi_{0xz}\phi_{2x} + \eta_0\phi_{0x}\phi_{2xz} + \phi_{0x}\phi_{3x} \\
& + \phi_{1x}\phi_{2x} + \eta_2\phi_{0z}\phi_0 + \eta_0\eta_1\phi_0^2 + \eta_0\eta_1\phi_{0z}^2 + \tfrac{2}{3}\eta_0^3\phi_0\phi_{0z} \\
& + \eta_1\phi_0\phi_{1z} + \eta_1\phi_{0z}\phi_{1zz} + \tfrac{1}{2}\eta_0^2\phi_{0z}\phi_{1z} + \eta_0^2\phi_0\phi_{1zz} + \tfrac{1}{2}\eta_0^2\phi_{0z}\phi_{1zzz} \\
& \eta_0\phi_{1z}\phi_{1zz} + \eta_0\phi_0\phi_{2z} + \eta_0\phi_{0z}\phi_{2zz} + \phi_{0z}\phi_{3z} + \phi_{1z}\phi_{2z}, \\
F_3(x,z,t) = {} & \eta_1\phi_{0tz} + \tfrac{1}{2}\eta_0^2\phi_{0t} + \eta_0\phi_{1tz} + \phi_{2t} + \eta_0\phi_{0x}\phi_{0xz} \\
& \phi_{0x}\phi_{1x} + \eta_0\phi_{0z}\phi_0 + \phi_{0z}\phi_{1z}.
\end{aligned}
$$

Full details of the calculations involved in the appendices can be obtained from the author or the JFM Editorial Office.

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